THE NECESSITY OF QUANTUM MECHANICS

Abruptness and abstruseness of the usual formulations

0) Historical necessity

1) Deforming the Poisson bracket

2) Quantum logic

3) Hardy's reasonable axioms

DEFORMING THE POISSON BRACKET

Dirac 1926 $\{f,g\}\to(i\hbar)^{-1}[\mathbf{f},\mathbf{g}]$

Weyl 1927 $\mathbf{g}=\dfrac{1}{4\pi^2}\displaystyle\int g(q,p)\mathrm{e}^{\mathrm{i}[\alpha(\mathbf{q}-q)+\beta(\mathbf{p}-p)]}\mathrm{d}q\mathrm{d}p\mathrm{d}\alpha\mathrm{d}\beta$

Wigner 1932 $\rho(q,p)=2\displaystyle\int \mathrm{d}q'\mathrm{e}^{2\mathrm{i}pq'/\hbar}\langle q-q'|\boldsymbol{\rho}|q+q'\rangle$

Moyal and Groenewold 1940s: $f*g=f\mathrm{e}^{(\mathrm{i}\hbar/2)(\overleftarrow{\partial}_q\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_q)}g \;\leftrightarrow\; \mathbf{fg}$

$$f*g-g*f=\mathrm{i}\hbar\{\{f,g\}\} \;\leftrightarrow\; [\mathbf{f},\mathbf{g}]$$

Quantum mechanics in phase space: $\dot{\rho}=\{\{H,\rho\}\}$

Vey 1975: Moyal bracket rediscovered as continuous one-parameter deformation of the Poisson bracket.

Lichnerowicz 1979 and Gutt 1979: This deformation is essentially unique!

DEFORMING THE POISSON BRACKET

Mathematical necessity of quantum mechanics

Transcendental necessity of Lie algebras of evolution (Poincaré)

Daniel Sternheimer:

A word of caution may be needed here. It is possible to intellectually imagine new physical theories by deforming existing ones …. Nevertheless such intellectual constructs, even if they are beautiful mathematical theories, need to be somehow confronted with physical reality in order to be taken seriously in physics. So some physical intuition is still needed when using deformation theory in physics.

QUANTUM LOGIC

Neumann's spectral theorem (1930): $G = \int_{-\infty}^{+\infty} \lambda \, P_\lambda \mathrm{d}\lambda$.

$P_\Lambda = \int_{\lambda\in\Lambda} P_\lambda \mathrm{d}\lambda$ is the orthogonal projector in Hilbert space associated with the proposition "The value of the observable G belongs to the subset Λ."

Neumann and Birkhoff 1936

Lattice of subspaces $\leftrightarrow$ Lattice of binary propositions

$a \leq b$	$a \wedge b$	$a \vee b$	$\bar{a}$	0	1
$A \subset B$	$A \cap B$	$A + B$	$A^\perp$	$\{\mathbf{0}\}$	$\mathcal{H}$

For finite dimension, this lattice is modular: If $a \leq c$, then $a \vee (b \wedge c) = (a \vee b) \wedge c$.

1) Is every orthocomplemented modular lattice isomorphic to the lattice of subspaces of a Hilbert space?
2) Are the axioms of such a lattice natural for Yes-No experiments?

QUANTUM LOGIC (question 1)

An irreducible complemented modular lattice is isomorphic to a projective geometry of the same dimension.

A projective geometry is isomorphic to the set of subspaces of a vector space over a field **K** with the operations $\subset$ and $+$.

If the lattice is orthocomplemented, the field **K** admits a * conjugation similar to complex conjugation and the vector space admits a sesqui-linear form similar to the Hermitian form of a Hilbert space.

C is *not* the only possibility.

Piron 1964: generalization to infinite dimension.

Mackey 1957: definition of states through probability of Yes for all Yes-No questions. Hence a state is a probability measure on the subspaces of a Hilbert space.

Gleason's theorem (1957): Such states must be described by a density matrix.

QUANTUM LOGIC (question 2)

Implication, meet, and join have natural definitions for finite dimension.

Modularity is more difficult to justify.

Piron: For finite dimension, it is equivalent to atomicity and weak modularity.

Piron: Weak modularity justified by non interference of measurement of b with measurement of a when $a \leq b$.

Piron: Atomicity (covering law) justified by existence of repeatable maximal measurements (defining pure states).

More problematic if infinite dimension.

Motivations: mathematical fertility (Neumann), most adequate language (Piron and Jauch), not so much insistence on necessity.

HARDY'S REASONABLE AXIOMS (2001)

Quantum theory is simply a new type of probability theory. Like classical probability theory it can be applied to a wide range of phenomena. However, the rules of classical probability theory can be determined by pure thought alone without any particular appeal to experiment (though, of course, to develop classical probability theory, we do employ some basic intuitions about the nature of the world). Is the same true of quantum theory? Put another way, could a 19th century theorist have developed quantum theory without access to the empirical data that later became available to his 20th century descendants? In this paper it will be shown that quantum theory follows from five very reasonable axioms which might well have been posited without any particular access to empirical data.

Hardy directly exploits a statistical definition of states, without the quantum-logic background.

HARDY'S REASONABLE AXIOMS

D_1: Quantity measurements have a maximal number N of discrete outcomes.
H_1: *Probabilities*. The relative frequency of a given outcome has a definite limit when the measurement is indefinitely repeated.

D_2: K is the minimal number of probabilities necessary to characterize the state of the system.
H_2: *Simplicity*. K takes the minimum value consistent with the axioms.

H_3: *Subspaces*. A system whose state is constrained to belong to an M-dimensional subspace … behaves like a system of dimension M.

H_4: *Composite systems*. A composite system consisting of subsystems A and B satisfies $N = N_A N_B$, $K' = K'_A K'_B$.

H_5: *Continuity*. There exists a continuous reversible transformation on a system between any two pure states of that system.

HARDY'S REASONABLE AXIOMS

These axioms lead to the matrix-density representation of states.

The simplicity axiom is not necessary (Dakić and Brukner).

All axioms except the subspace axiom can be justified by considerations of correspondence.

The continuity axiom can be replaced by information-theoretic axioms, for instance the assumption that two-level systems carry one bit of information. Physical necessity seems lost in this process.

CONCLUSIONS

1) Starting from classical mechanics, deformation of Poisson bracket shows mathematical, perhaps transcendental necessity of the whole quantum mechanics (including the expression of the Hamiltonian).

2) Starting from a very basic notion of experimentation, quantum logic shows necessity of **K***-space structure for finite dimension. It requires difficult math. It does not exclude **K** fields differing from **C**. It does not specify the Hamiltonian.

3) Starting from statistics of discrete measurements, Hardy axiomatics leads to the density-matrix representation of states. It implicitly assumes some correspondence with classical description, and it does not specify the Hamiltonian.

Sources of quantum weirdness: negative probabilities in the deformation approach, incompatible measurements in quantum logic, blending of discrete measurement outcomes with continuous possibilities of measurements in Hardy axiomatics.